Approximations of invasion phases and invasion times for populations

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Groupe de travail PEIPS
11 July 2023

1. Introduction
   - Point process
   - Invasion model
2. Results
   - Statements
   - Hitting times approximations

Point process : definitions

A **point process** Z is :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i\in\mathbb{N}} \delta_{T_i}$

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A process λ is the **stochastic intensity** of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b])|\mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

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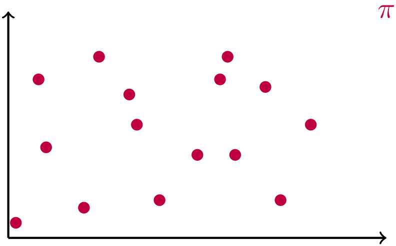
Interpretation :

- T_i = random times of "events"
- λ = random rate of "events"

Thinning

π Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt.dz$



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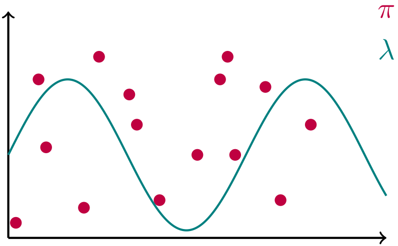

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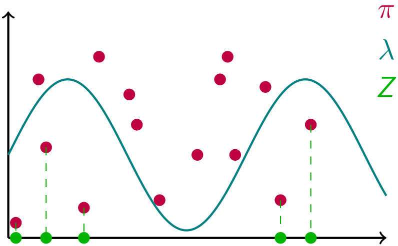

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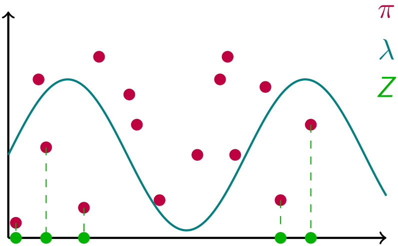

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- $d(N_t^K/K)$ individual **death** rate

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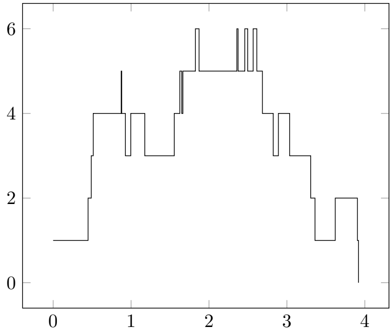
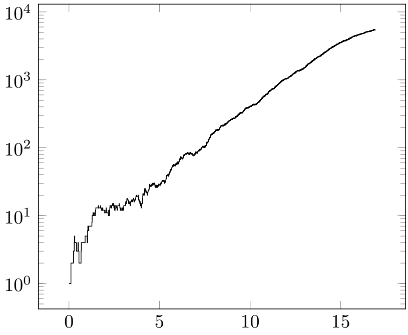
Two scenarios :

- extinction $\longrightarrow \exists t, N_t^K = 0$
- invasion $\longrightarrow \exists t, N_t^K = \Theta(K)$





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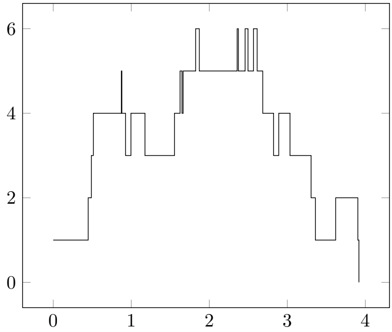

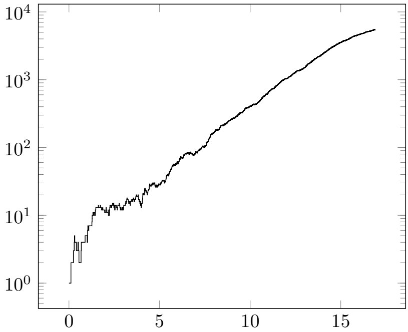

Assumption : supercritical regime $r_0 := b(0) - d(0) > 0$

Problematic

Model : $N_t^K = N_0^K + Z_t^{b,K} - Z_t^{d,K}$ with $r_0 := b(0) - d(0) > 0$

Question : what is the **detection time** of the invasion ?

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Our goal : develop the asymptotic expansion of $T_{1\to K}^K$

Heuristics

$$N_t^K = N_0^K + \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \leq N_{s-}^K b(N_{s-}^K/K)\}} d\pi_b(s,z) - \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \leq N_{s-}^K d(N_{s-}^K/K)\}} d\pi_d(s,z)$$

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Branching approximation while $N_t^K \ll K$ (i.e. $N_t^K/K \approx 0$)

$$N_t^K \approx Z_t = Z_0 + \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \le Z_{s-} b(0)\}} d\pi_b(s,z) - \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{z \le Z_{s-} d(0)\}} d\pi_d(s,z)$$

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ODE approximation if $N_{t_0}^K = \Theta(K)$ (i.e. $N_{t_0}^K/K = \Theta(1)$)

$$\frac{N_t^K}{K} \approx x_t^K = x_{t_0}^K + \int_{t_0}^t x_s^K \left(b(x_s^K) - d(x_s^K) \right) ds$$

Formal results

Branching approximation :

ODE approximation :

Hitting time approximation :

Formal results

Branching approximation : if $\xi_K \ll K/\ln K$

$$\mathbb{P}\left(\sup_{t \leq T^K_{1\to\xi_K}} \left| \frac{N^K_t}{Z_t} - 1 \right| > \eta \middle| \{\inf Z > 0\} \right) \underset{K\to\infty}{\longrightarrow} 0$$

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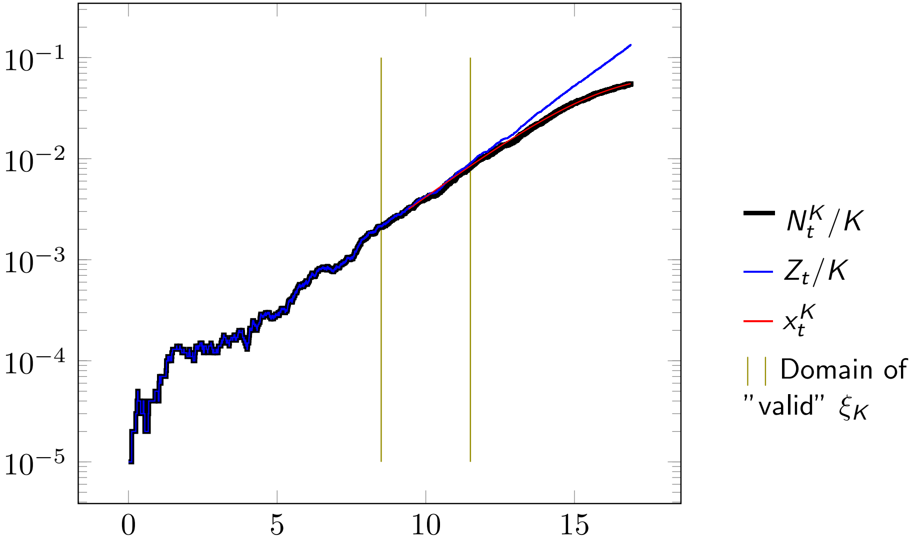
Hitting time approximation : conditionally on $\{\inf Z > 0\}$

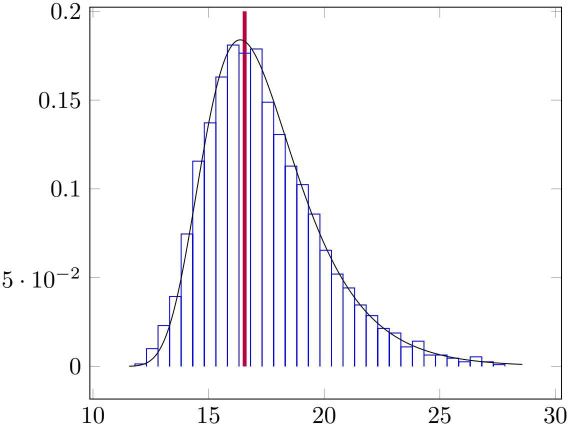
$$T^K_{1\to K} = \frac{1}{r_0}\ln K + \frac{1}{r_0}\ln(1/W) + \int_0^1 \frac{1}{x}\left(\frac{1}{b(x)-d(x)} - \frac{1}{r_0}\right)dx + o(1)$$

with $W \sim \mathcal{E}\left(r_0/b(0)\right)$ and $r_0 = b(0) - d(0)$

Simulation : processes approximations



Simulations : hitting times approximation



— empirical distribution of $T^K_{1\to K}$

— density of : $\frac{1}{r_0}\ln K + \frac{1}{r_0}\ln(1/W) + \int_0^1 \frac{1}{x}\left(\frac{1}{b(x)-d(x)} - \frac{1}{r_0}\right)dx$

Comparison with literature

Results of [Barbour, Hamza, Kaspi, Klebaner (2015)]

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Thank you for your attention !

Questions ?